WEIBULL DISTRIBUTION AND ANALYSIS: 2019

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ABSTRACT
Weibull Analysis, while initially met with skepticism, is now used across many disciplines in reliability and survival analysis. This paper will provide an overview of the Weibull distribution, its variables, the types of data required, and the interpretations that can be drawn from a Weibull distribution. The appendix will provide a tutorial on Weibull Analysis in Minitab.

Key words: Weibull distribution, reliability analysis, survival analysis, hazard function, Minitab.

I. INTRODUCTION: THE WEIBULL DISTRIBUTION

The Weibull distribution was invented by Swedish engineer Waloddi Weibull (1887-1979) in 1937. He published his paper on the subject in 1951. The distribution describes cumulative failure rates and is often used in survival or reliability analysis of products.

In contrast with binomial or Poisson distributions, the Weibull distribution can be reliable even with a very small sample size. This is very useful, as failures can be costly and dangerous, so large failure samples cannot be obtained.

The cumulative distribution function (CDF) of the Weibull distribution is as follows, were η represents the characteristic life, or the age at which 63.2% of units will have failed, and β represents the slope of the best-fit line.

$$F(t) = 1 - e^{-(t/\eta)^{\beta}}$$

The derivative of the CDF is the PDF, and interestingly, it resembles a normal distribution:

$$\frac{dF(t)}{dt} = (\beta/\eta)^{\beta-1} e^{-(t/\eta)^{\beta}}$$

Note that the mean-time-to-failure (MTTF), or average life of the unit, is only equal to η when β=1. Otherwise, if β > 1, MTTF will be less than η, and if B < 1 will be greater than η. The following equation can be used to find MTTF:

$$\text{MTTF} = \eta\Gamma(1+1/\beta)$$

The Meanings of β Values

A shallow slope, where β < 1, indicates infant mortality. This can be due to factors such as inadequate burn-in or stress testing, or defective/misassembled parts. In the case of infant mortality, overhauls and inspections of units are not cost effective. When β = 1, random failures are indicated, and again, overhauls and inspections are recommended.

If β > 1, wear-out failures are indicated. In this case, overhauls and inspections are cost effective, as the life of the unit can be easily predicted and overhauls or inspections can be scheduled at a time when the probability of failure is appropriate.

A high β is ideal in a Weibull curve, as a steep line indicates a predictable age of failure; however, as will be discussed in Section VI, steep plots can also obscure issues in the data.

II. DIRTY DATA

Several data characteristics can impede the Weibull distribution's ability to effectively characterize the lifespan or reliability of a unit. These data are known as deficient or "dirty" data, and include the following attributes (Abernathy 1-9):

- Censored or suspended data

Censored or suspended data are data that are not included in the Weibull plot. These may include units that did not fail during the monitoring period or failures of a different failure mode than the one being studied. Although they are not plotted, they still must be included in statistical analyses.

- Mixtures of failure modes

Sometimes units can fail in different ways, i.e. different parts of a machine might break. In this case, the failure data might be distributed along different lines in the plot. In this situation, root cause analysis should be performed and then the different failure causes can be analyzed separately. Multiple failure modes will be discussed in Section IV.

- Failed units not identified
- Inspection & coarse data

If data are collected during weekly or monthly inspections, the precise failure time is often not recorded, and this will alter the Weibull distribution.

- Suspension times or ages missing
- No failure data
- Nonzero time origin

The phenomenon of nonzero t_0 will be discussed in Section VI.

- Extremely small samples

Small sample sizes are acceptable for engineering prediction purposes, but for accurate statistical analysis, larger samples are needed.

- Early data missing
- Data plots with curves and doglegs

This can often be explained by data that fit one of the above characteristics. Curved data are often related to an issue in time origin, and doglegs often indicate multiple failure modes. This will be discussed further in Section VI.

How is Age Measured?

Depending on the type of unit being analyzed, different age-related factors will lead to wear-out failures. Age can be measured in time (hours, days, months), but also in on/off cycles, cycles at high stress or temperature, miles travelled, etc. If the appropriate age metric is not obvious, plot the different options and see which one fits the data best.

III. WEIBULL IN ELECTRONICS

Failure of solder joints in electronics is often caused by thermal cycling (at approximately 125-130°C) or drop shocks. Thermal cycling damages solder joints because as the heat increases, different parts reach different temperatures and expand at different rates, and thus the solder balls are constantly stressed with this expansion. The phenomenon of drop shock failures began with the rise of personal technology. Handheld and personal devices are extremely susceptible to being dropped, and because of this, electronics must be resistant to drop shock. Manufacturers test both temperature and drop shock resistance before the release of products to prevent failure after purchase.

An example of solder joint failure analysis is a comparison of different lead-free solders and tin-lead solder. As lead solder was banned in the EU in 2006, lead-free solders have been increasingly implemented in recent years. SAC105, SAC305, and SACM were compared with SnPb (tin-lead solder) by Liu et al in 2009 (Liu et. al.). Note: SAC105 indicates 1% silver, 0.5% copper, and the remaining percentage tin. SACM is a lead-free solder alloy including manganese in addition to tin, silver, and copper.

Figure 1 shows Weibull plots for each material. SACM is shown to have the highest η value (2087.15 cycles) and a decent β value (5.52). Based on these data, we would say that SACM is the best alloy for soldering.

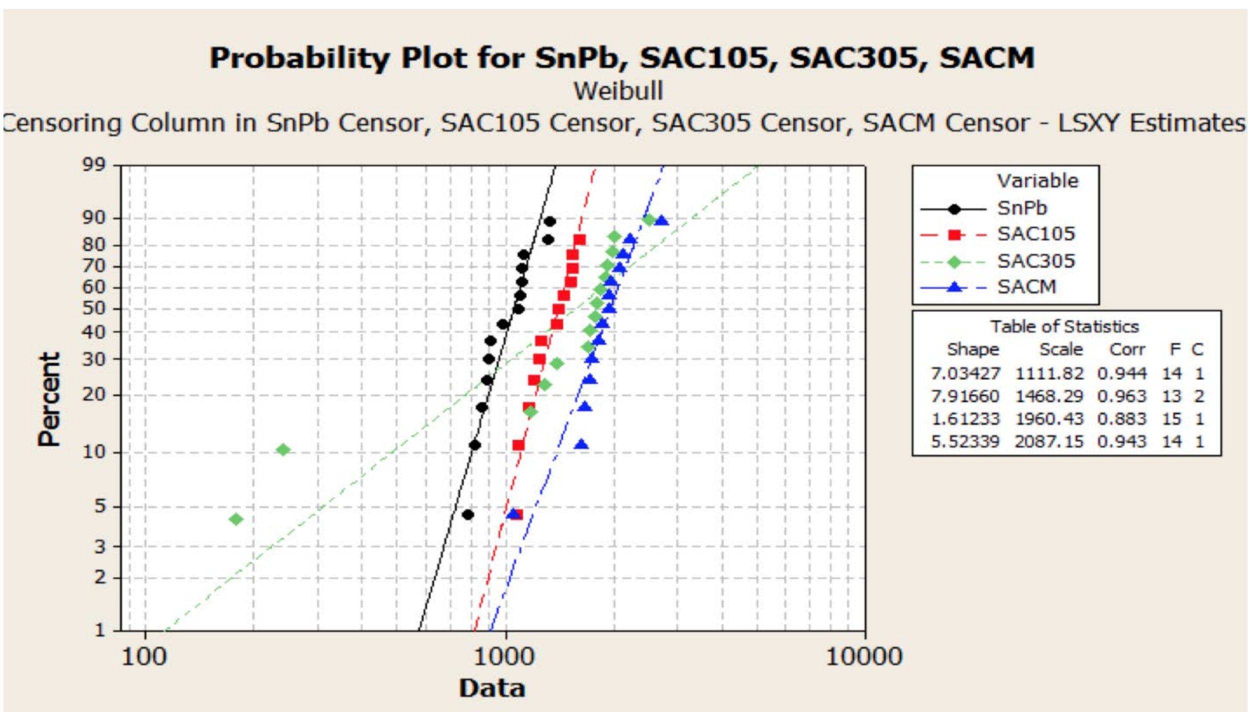

Figure 1: Weibull Probability Plot for Four Solder Alloys (data from Liu et.al.)

Note that SAC305 has a much lower β than the other alloys, but this is because of two infant mortality points that affect the slope tremendously. Without the early fails, we can see that the shape of SAC305 is similar to the other alloys. This demonstrates that the inclusion of early fails distorts the shape of the curve.

In this situation, the test should be suspended after the early first fails and root cause analysis should be performed. If the units are defective, the test should be repeated without defective units to obtain a more accurate β and η.

IV. MULTIPLE FAILURE MODES

Multiple failure modes occur when different problems cause the unit to fail, often at different times in the unit's life. Defective or misassembled items will appear as early failures and represent a different failure mode from later wear-out failures.

A bathtub-shaped hazard plot indicates both infant mortality and wear-out failure. The shape of this hazard curve is seen in Figure 2.

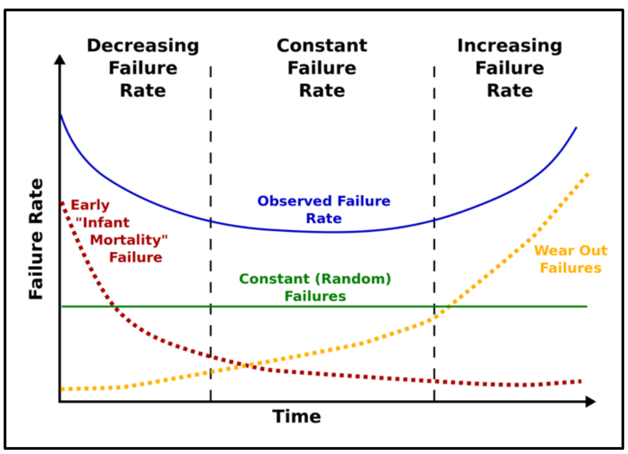

Figure 2: Bathtub-Shaped Hazard Function

On a Weibull plot, these data will show a shallow slope near the origin with a sharp upturn in slope at the age when infant mortality gives way to wear-out failure, as shown in Figure 3. This type of Weibull plot is called a Classic Bi-Weibull.

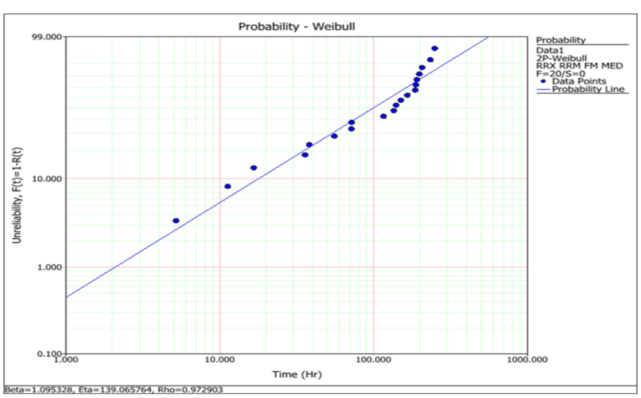

Figure 3: Classic Bi-Weibull Plot

To prevent infant mortality in consumer products, companies administer stress tests to electronics to avoid customer reception of a product that will fail immediately.

V. THE HAZARD FUNCTION

The hazard function can be a useful way to analyze failure risk. This function represents the chance that a unit will fail at a specific point in time, whereas the Weibull CDF maps the cumulative percentage of units that will have failed at each point in time. The hazard function can be plotted as follows:

$$h(t) = (\beta/\eta)(t/\eta)^{(\beta-1)}$$

The hazard function plots as horizontal for β = 1, indicating that failure is random and the chance of failure remains constant along a unit's life. The hazard function curves down for β < 1, indicating infant mortality, meaning that the chance of failure decreases as time progresses. The hazard function curves up for β > 1, indicating wear-out failure, meaning that the chance of failure increases as a unit ages.

VI. GOODNESS OF FIT AND THE THREE-PARAMETER WEIBULL

The correlation coefficient "r" indicates the strength and direction of a linear correlation. To analyze the "goodness of fit" of a Weibull plot, best practice is to use the r value given by the program used to analyze the data, and then compare the r value to the critical correlation coefficient (CCC). The CCC can be found by performing 10,000 Monte Carlo simulation trials, ranking them in order from largest to smallest r-value, and choosing the 1000th value. In other words, the CCC is the 90th percentile of r values across Monte Carlo trials. The same test can be conducted with r^2. If the r-value from the Weibull plot is greater than the CCC, a good fit is indicated. If r < CCC, a poor fit is indicated.

Computer programs can provide the Pve, or p value of r and r^2, and this is the most accurate figure to measure goodness of fit. If the Pve < 10%, this indicates a bad fit. A Pve > 10% indicates a good fit.

Three-Parameter Weibull

If failure data form a curved plot, this most often indicates that the time origin (t_0) has been inappropriately set. Age of possible failure does not always start at zero. For example, "a bearing failure due to spalling or unbalance cannot occur without bearing rotation inducing enough damage to fail the bearing" (Abernathy 3-8). In this case, it is impossible for failure to occur at t=0.

Age of possible failure can also start at a negative time value: for example, materials such as rubber and chemical solutions age in storage, and can begin deteriorating before the unit in question is assembled. In this case, failure can occur before t=0.

Data curved downward often indicate a positive t_0, while data curved upwards often indicate a negative t_0. Doglegs and corners indicate mixed failure modes, and vertical columns of points indicate that data are coarse and have been collected at set inspection dates rather than at true failure points.

Note that steep curves can obscure bad data. Origin problems, manifested in curves and doglegs, can disappear when β is high. As a general rule, the plot should be scrutinized if $\beta>6$. In this case, the curve may appear to fit the line well, but Pve values may indicate a poor fit.

For these issues, a three-parameter Weibull equation can be useful:

$$F(t) = 1 - e^{-[(t-t0)/\eta]^\wedge\beta}$$

For a three-parameter Weibull to be effective, the following must be true:

- The failure data must form a curved plot
- There must be a physical explanation for shift in origin (e.g. chemicals degrading in storage)
- The data must contain at least 21 failures
- The correlation coefficient p should increase when three-parameter Weibull is used vs. two-parameter Weibull

Sometimes, downward-curving plots can be better fit by a log normal distribution, rather than a Weibull distribution. An example of this is progressive deterioration. For example, vibration loosens parts, and these loosened parts in turn increase vibration even more (Abernathy 3-12). For these cases, use the Pve value given by a computer program to determine whether a Weibull or log normal distribution is a better fit.

APPENDIX: WEIBULL ANALYSIS IN MINITAB

This section will provide a tutorial in Weibull analysis using Minitab. The data used will be the solder data from Section III of this paper (Liu et al.).

First, copy and paste the data into Minitab, and then select the following menu items, as seen in Figure 4: Stat > Reliability/Survival > Distribution Analysis (Right Censoring) > Parametric Distribution Analysis. Here we use only the data for SAC305.

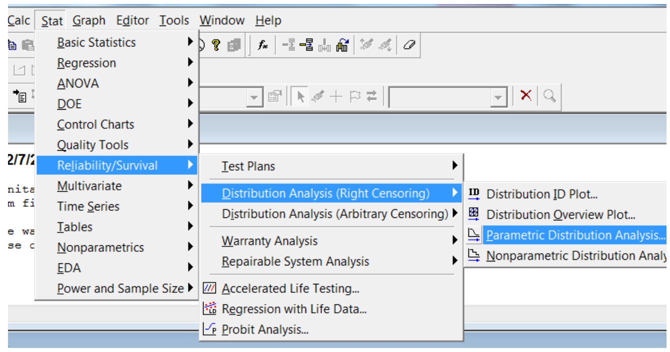

Figure 4: Minitab, First Menu Selections

Two windows will appear. In the "Parametric Distribution Analysis-Right Censoring" window, make sure that the desired variables are listed and that the assumed distribution is set to "Weibull." For this tutorial, we will first select only the SAC305 data. In the "Parametric Distribution Analysis: Estimate" window, Minitab automatically selects Maximum Likelihood as the estimation method. Another option is Least Squares—these two methods will yield slightly different results. After pressing "OK" on each of the two windows, the probability plot will appear (see Figure 5):

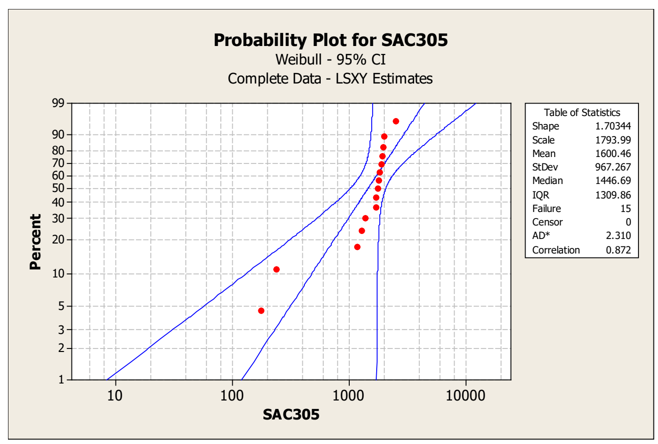

Figure 5: Minitab Probability Plot for SAC305

We can also obtain a number of other useful plots by selecting "Distribution Overview Plot" instead of "Parametric Distribution Analysis" in the final menu in Figure 4. Again, two windows will appear, and after selection of the proper parameters, the Distribution Overview Plot will appear, including four graphs (see Figure 6). Here, we can see the PDF, the Weibull plot, the Survival Function, and the Hazard Function of the SAC305 data. The Survival Function can be used to determine the life expectancy of a unit, and the Hazard Function, as discussed in Section V, provides the likelihood of failure at each specific point in time.

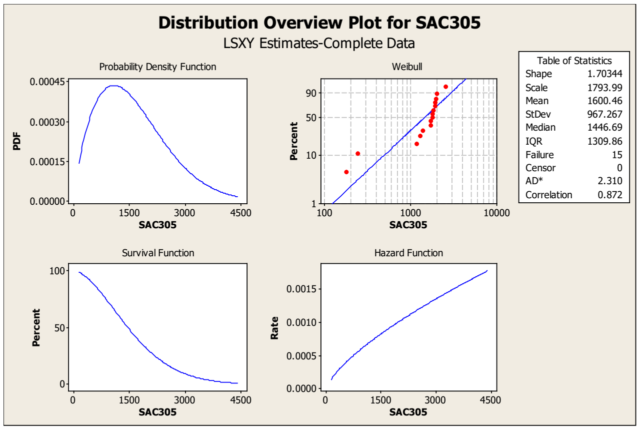

Figure 6: Minitab Distribution Overview Plot for SAC305

To obtain the comparison Weibull Plot, Figure 1 in Section III, the same steps are followed in Minitab, except in the "Parametric Distribution Analysis-Right Censoring" window, multiple data sets must be selected (see Figure 7):

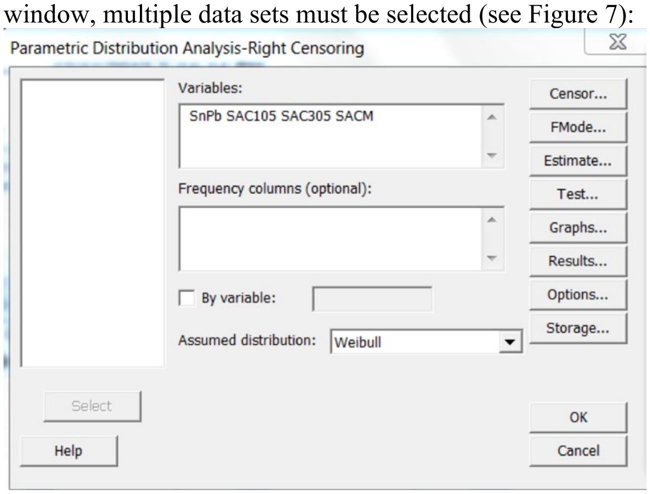

Figure 7: Minitab Multiple Variable Selection

In addition, the data must be censored as there are units that did not fail during the tests. The following window will appear, and censored columns can be selected (see Figure 8):

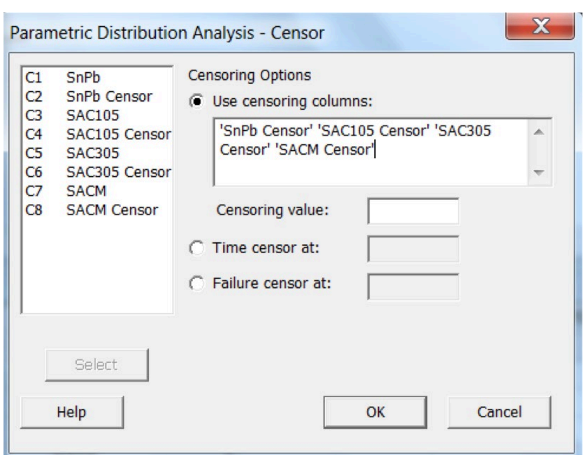

Figure 8: Minitab Censor Column Selection

After following these steps, the Probability Plot for all of the variables will appear (see Figure 9, or Figure 1 in Section III):

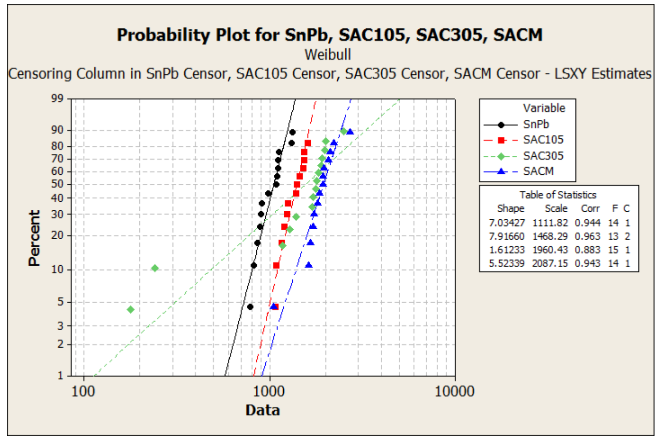

Figure 9: Minitab Probability Plot for All Four Variables

Using this plot, we can compare the reliability of the four types of solder. This comparison can be found in Section III.

REFERENCES

Abernethy, Robert B. 2004. The New Weibull Handbook: Reliability and Statistical Analysis for Predicting Life, Safety, Survivability, Risk, Cost, and Warranty Claims (5e). North Palm Beach, FL.

Liu, Weiping et. al. 2009. "Achieving High Reliabiliy Low-Cost Lead-Free SAC Solder Joints via Mn or Ce Doping. ECTC.

IMAGE SOURCES:

Figure 2: https://en.wikipedia.org/wiki/Bathtub_curve

Figure 3: http://help.synthesis8.com/weibull_alta8/mixed_weibull_analysis.htm

All other images are screenshots created by the authors while using Minitab.